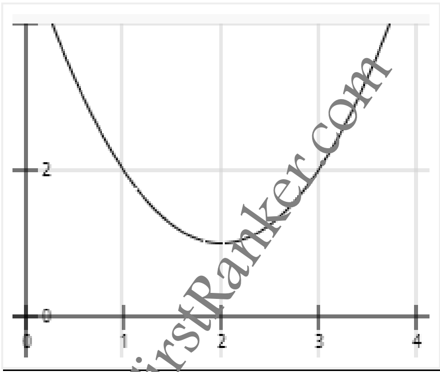
Maxima and Minima

Maxima and minima is a very important chapter as far as CAT, XAT, SNAP etc exams are concerned. Before you read this lesson, read Slope of polynomial

1. Find the minimum value of $x^2 - 4x + 5$.

We know that graph of this equation is concave up. As you can see from the graph that it won't touch the x-axis so it does not have any real root. But we can find, where this graph attains its minima we can't find maxima.



Differentiating the given function we get 2x - 4.

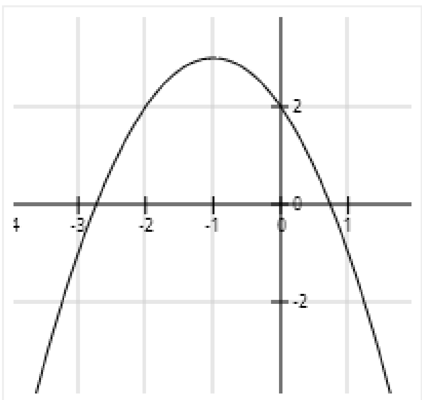
We equate this expression to zero to find where minima exists.

2x - 4 = 0 and x = 2.

Substituting in the given expression we get $2^2 - 4.2 + 5$ = 1.

2. Find the maxima value of $2 - 2x - x^2$.

x^2 coefficient is negative. As this graph is concave down, it has maxima. We can't find the minimum



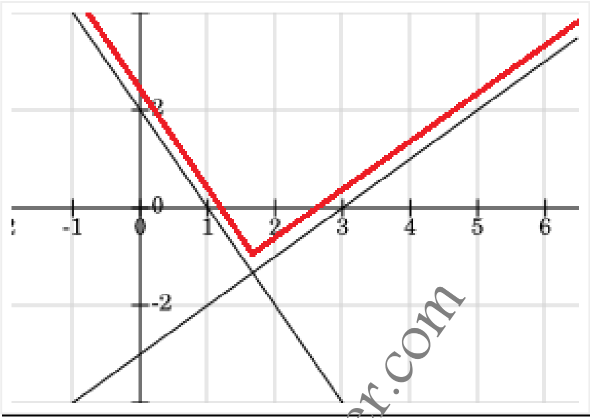
Differentiating the given expression we get, - 2 - 2x.

Equating to zero, we get x = -1

So at x = -1 it attains maxima which is equal to $2 - 2(-1) - (-1)^2 = 3$

3. y = max (2-2x, x - 3) then find the minimum value of this function.

The given function is a combination of two linear equations. 2 - 2x is a downward sloping line, and x - 3 is a upward sloping line. As y is defined as max of these two equations, y can be represented as the graph noted with red line. That is upto some point in between 1 and 2, it decreases, and starts increasing after that point. so this graph attains minimum where these two lines intersect.



Equating, 2 - 2x = x - 3

We get x = 5/ 3

So minimum value can be obtained by substituting x value in any of these linear equations. 2 - 2(5/3) = -4/3.

Three very important Rules:

Rule 1: For positive variables, if the sum of the variables is a constant, the product of the variables will be maximum when all the variables are equal.

Eg: If a + b + c = 21, find the maximum value of abc.

Here sum of the variable is constant. So product will be maximum, when all the three variables are equal i.e., 3a = 21, a = 7. So product = 7 x 7 x 7 = 343

Rule 2: For positive variables, if the product of the variables is a constant, the sum of the variables will be minimum when all the variables are equal

Eg: Find the minimum value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$

Here the product of the variables = $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a} = 1$

So given sum is minimum when all are equal $\frac{a}{b} = 1, \frac{b}{c} = 1, \frac{c}{a} = 1$

So sum = 1 + 1 + 1 = 3.

$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$

Rule 3: For positive variables, Arithmetic Mean (AM), is always greater than Geometric Mean (GM) i.e., $A.M \geq G.M$

Eg: If xy = 16, then find the minimum value of x + y.

AM of x, y = $\frac{x+y}{2}$

GM of x, y = $\sqrt{(x.y)}$

from AM GM rule $\frac{x+y}{2} \geq \sqrt{(x.y)}$

Substituting xy = 16, we get $\frac{x+y}{2} \geq 4$

Or $x + y \geq 8$

Other Examples:

1. Find the greatest value of $a^2.b^3.c^4$ subject to the condition a+b+c=18

Sol: Though sum of the variables are constant in this question, directly we cannot apply the rules learned above.

We have to modify the given expression to suit the above rules

Let Z = $a^2.b^3.c^4$

Z = $2^2.3^3.4^4.(\frac{a}{2})^2.(\frac{b}{3})^3.(\frac{c}{4})^4$

[any question of this type, we modify a^p as $(\frac{a}{p})^p$ so on and multiply with suitable powers to make it equal to original equation]

Z will have the maximum when $(\frac{a}{2})^2.(\frac{b}{3})^3.(\frac{c}{4})^4$ is maximum.

But $(\frac{a}{2})^2.(\frac{b}{3})^3.(\frac{c}{4})^4$ is a product of 2+3+4=9 factors whose sum = $2(\frac{a}{2})+3(\frac{b}{3})+4(\frac{c}{4})$ = a+ b + c = 18

$(\frac{a}{2})^2.(\frac{b}{3})^3.(\frac{c}{4})^4$ will be maximum if all the factors are equal. i.e., if $\frac{a}{2} = \frac{b}{3} = \frac{c}{4} = \frac{a+b+c}{9} = \frac{18}{9} = 2$

So maximum value of Z = $2^2.3^3.4^4.(2)^2.(2)^3.(2)^4 = 2^{19}.3^3$

Alternate method:

The greatest value of $a^m.b^n.c^p$, when m, n, p being +ve integers, a+b+c is constant is given by

$m^m.n^n.p^p......(\frac{a+b+c+...}{m+n+p+...})^{m+n+p+..}$

By applying above concept: $2^2.3^3.4^4.(\frac{18}{9})^9 = 2^{19}.3^3$

2. If 2x+3y=7; find the greatest value of $x^3.y^4$

Solution: Let Z = $x^3.y^4$

[we change the original function by taking $(\frac{2x}{3})^3$ instead x^3 and $(\frac{3y}{4})^4$ instead of y^4]

So Z = $x^3.y^4$ = $(\frac{3}{2})^3(\frac{4}{3})^4(\frac{2x}{3})^3(\frac{3y}{4})^4$

But $(\frac{2x}{3})^3(\frac{3y}{4})^4$ is a product of 3 + 4 = 7 factors, whose sum = $3(\frac{2x}{3})+4(\frac{3y}{4})$ = 2x + 3y = 7

Therefore; $(\frac{2x}{3})^3(\frac{3y}{4})^4$ will be maximum if all the factors are equal

i.e., $\frac{2x}{3} = \frac{3y}{4} = \frac{2x+3y}{3+4} = \frac{7}{7} = 1$

So maximum value of Z = $(\frac{3}{2})^3(\frac{4}{3})^4(1)^3(1)^4$ = $\frac{27}{8} \times \frac{256}{81} = \frac{32}{3}$

Alternate method:

We partial differentiate the given function w.r.t x and then with y and find the ratio. Also we partial differentiate 2x+3y = 7 w.r.t x and then with y and find the ratio. Now we equate these two ratio's and find y value interms of x.

$\Rightarrow \frac{3x^2y^4}{4y^3x^3} = \frac{2}{3}$

$\Rightarrow \frac{3y}{4x} = \frac{2}{3} \Rightarrow \frac{y}{x} = \frac{8}{9}$

$\Rightarrow y = \frac{8}{9}x$

Substituting in 2x + 3y = 7 we get x = $\frac{3}{2}$

Now we find value of y as $\frac{4}{3}$

So maximum value of $x^3.y^4$ = $(\frac{3}{2})^3.(\frac{4}{3})^4 = \frac{32}{3}$

3. If x, y , z are positive reals such that $x^3y^2z^4 = 7$ then find the minimum value of $2x+5y+3z$

We modify the product to apply AM GM rule.

Consider the product $(\frac{2x}{3})^3(\frac{5y}{2})^2(\frac{3z}{4})^4$

Above is the product of nine quantities.

Apply AM $\geq$ GM

$$\Rightarrow \frac{(3.\frac{2x}{3}+2.\frac{5y}{2}+4.\frac{3z}{4})}{3+2+4} \geq \{(\frac{2x}{3})^3.(\frac{5y}{2})^2.(\frac{3z}{4})^4\}^{1/9}$$

$$\Rightarrow \frac{(3.\frac{2x}{3}+2.\frac{5y}{2}+4.\frac{3z}{4})}{3+2+4} \geq \{(\frac{2}{3})^3.(\frac{5}{2})^2.(\frac{3}{4})^4.x^3y^2z^4\}^{1/9}$$

$$\Rightarrow 2x+5y+3z \geq 9\{\frac{8}{27}\cdot\frac{25}{4}\cdot\frac{81}{256}.7\}^{1/9}$$

$$\Rightarrow 2x+5y+3z \geq 9\{\frac{525}{2^7}\}^{1/9}$$

4. Find the maximum value of $(7-x)^4(2+x)^5$ when x lies between - 2, 7.

To apply any of the above said rules, we first consider that the given terms are positive or not. 7-x, 2+x both are positive between -2, 7

We have to find max. value of $(7-x)^4(2+x)^5$ or A^4B^5 where A + B = 9.

It will be maximum if $(\frac{A}{4})^4(\frac{B}{5})^5$ is maximum

Their sum is $4(\frac{A}{4})+5(\frac{B}{5})$ = A + B = 9

For max.product $\frac{A}{4}=\frac{B}{5}=\frac{A+B}{4+5}=\frac{9}{9}=1$

So A = 4 and B = 5

Max. product is $4^4 5^5$

Alternate Method:

We know that

The greatest value of $a^m.b^n.c^p$, when m, n, p being +ve integers, a+b+c is constant is given by

$$m^m.n^n.p^p......(\frac{a+b+c+...}{m+n+p+...})^{m+n+p+..}$$

Therefore max value of the above = $4^4 5^5(\frac{7-x+2+x}{4+5})^{4+5} = 4^4 5^5(\frac{9}{9})^{4+5} = 4^4 5^5$